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The photon energy spectrum in $B \to X_s + \gamma$ in perturbative QCD through $\mathcal{O}(\alpha_s^2)$

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We derive the dominant part of the $\mathcal{O}(\alpha_s^2)$ correction to the photon energy spectrum in the inclusive decay $B \to X_s + \gamma$. The detailed knowledge of the spectrum is important for relating the theoretical calculations of the $B \to X_s + \gamma$ decay rate and the experimental measurements where a cut on the photon energy is applied. In addition, moments of the photon energy spectrum are used for the determination of the b-quark mass and other fundamental parameters of heavy quark physics. Our calculation reduces the theoretical uncertainty associated with uncalculated higher orders effects and shows that, for $B \to X_s + \gamma$, QCD radiative corrections to the photon energy spectrum are under theoretical control.

I. INTRODUCTION

The process $B \to X_s + \gamma$ plays an important role in particle physics. Within the Standard Model, it is loop-induced; in many extensions of the Standard Model the decay rate of $B \to X_s + \gamma$ receives sizable contributions from virtual, yet undiscovered particles. Comparison of the experimentally measured rate for $B \to X_s + \gamma$ with the theoretical expectations puts stringent constraints on various new physics scenarios.

Thanks to BaBar, Belle and CLEO experiments, high quality data on $B \to X_s + \gamma$ is available at present [1, 2, 3, 4]. Due to the large irreducible background at small photon energies, all measurements are performed with a lower cut on the photon energy $E_\gamma > E_{\rm cut}$. Until recently, $E_{\rm cut} \sim 2.0$ GeV was routinely used. Last year Belle [1] presented precise measurements of the branching fraction and the first two moments of the photon energy spectrum with the cut as low as $E_{\rm cut} = 1.815$ GeV. This year, preliminary results with $E_{\rm cut} \geq 1.9$ GeV were reported by BaBar collaboration [2].

The theoretical understanding of the decay $B \to X_s + \gamma$ in the Standard Model is currently quite robust. The total decay rate is known through next-to-leading order (NLO) in perturbative QCD [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16] and the process of calculating it through next-to-next-to-leading order (NNLO) is under way [17, 18, 19, 20, 21]. However, because the experimental measurements require a cut on E_γ, even a perfect knowledge of the total, fully inclusive decay rate is only useful if the the photon energy spectrum is understood well enough to relate a measurement of the branching fraction with $E_\gamma > E_{\rm cut}$ to the theoretically known inclusive total rate. In addition, the detailed knowledge of the photon energy spectrum has its own merit. Since the photon energy spectrum is largely insensitive to contributions from beyond the Standard Model physics, it is used to study strong interaction physics in heavy flavor decays. In particular, the value of the b-quark mass, the average kinetic energy of a b quark in a B-meson and the importance of the shape function for different values of the cut on the photon energy can be investigated.

The theoretical description of the radiative b-decays is based on an effective theory approach where all particles heavier than the b quark are integrated out. The full effective Lagrangian that is used in the calculation of radiative decays of B-mesons can be found in [12, 22]. It is well established [23, 24] that the dominant contribution to the photon energy spectrum comes from the local operator $\hat{O}_7 \sim \bar{s}\sigma_{\mu\nu} b F_{\mu\nu}$, while for the computation of the total decay rate, other operators are also important.

The photon energy spectrum in $b \to X_s + \gamma$ is currently known through $\mathcal{O}(\alpha_s)$. The effects of higher order QCD corrections are traditionally studied [24] in the so-called Brodsky-Lepage-Mackenzie (BLM) approximation [25]; the BLM corrections are both the easiest to compute and often provide the dominant part of the $\mathcal{O}(\alpha_s^2)$ corrections. In flavor physics, where typical energy scales are relatively low and hence the strong coupling constant is large, the BLM corrections can be significant and their naive application could lead to inflated theoretical uncertainties. A consistent application of the BLM corrections in B-decays was developed in a series of papers [26, 27]. Similar approach to the photon energy spectrum in $B \to X_s + \gamma$ has been initiated in [28] and later elaborated upon in [29, 30] where, in particular, the BLM corrections have been resummed to all orders in the strong coupling constant. Among other things, this approach requires a careful separation of perturbative and non-perturbative effects and understanding the intricate interplay between them.

Additional theoretical information on the shape of the photon energy spectrum or on the integrated spectrum in the presence of a lower cut on the photon energy,

can be obtained from the universality of the soft and collinear gluon radiation [31]. Such an approach can be used to predict large logarithms $\sim \ln(E_{\rm max} - E_{\rm cut})$, where $E_{\rm max}$ is the maximal energy allowed for the photon in $B \to X_s + \gamma$. At present, such computations are performed with next-to-next-to-leading logarithmic accuracy [32] (see also [33]). On general grounds it is clear that such an approach is applicable only for $E_{\rm cut}/E_{\rm max} \approx 1$ and not for moderate or low values of the cut on the photon energy.

Further improvement in the theoretical description of the photon energy spectrum in $b \to X_s + \gamma$ requires $\mathcal{O}(\alpha_s^2)$ corrections beyond the BLM approximation and it is the purpose of this paper to provide them. We restrict ourselves to the contribution generated by the operator $\hat{O}_7$ since, as we pointed out above, this operator provides the leading contribution to the photon energy spectrum.

The paper is organized as follows. In Section II we present the $\mathcal{O}(\alpha_s^2)$ correction to the photon energy spectrum in $b \to X_s + \gamma$ and discuss its derivation. In Section III we study the properties of the spectrum and compare it with known partial results. In Section IV we consider the effect of our result on the evaluation of the first moment of the photon energy spectrum and on the branching ratio $\mathrm{Br}[B \to X_s\gamma]$ in the presence of a lower cut on the photon energy. Finally, we present our conclusions.

II. THE PERTURBATIVE EVALUATION OF THE PHOTON SPECTRUM

We consider the decay of an on-shell b-quark into a photon and any hadronic state containing a strange quark. We assume that this decay is mediated by the effective operator $\hat{O}_7$:

$$\hat{O}_7 = \frac{e}{32\pi^2}\overline{m}\; \bar{s}\sigma_{\mu\nu}F^{\mu\nu}(1+\gamma_5)b. \tag{1}$$

Here, $F_{\mu\nu}$ is the electromagnetic field strength tensor, $e = \sqrt{4\pi\alpha}$, where α is the fine structure constant evaluated at zero momentum transfer [34] and $\overline{m}$ is the $\overline{\rm MS}$ b-quark mass evaluated at the b-quark pole mass $\overline{m} = \overline{m}(m_b)$. The photon energy spectrum is parameterized in terms of the variable:

$$z = \frac{2p_\gamma p_b}{m_b^2} = \frac{2E_\gamma}{m_b}, \tag{2}$$

where the last equality is valid in the b-quark rest frame. When the photon energy spectrum is studied within perturbation theory, the kinematically allowed range for the photon energy implies[1] $0 \le z \le 1$. For our calculation, we neglect all quark masses other than the mass of the b quark and use dimensional regularization for ultraviolet, infra-red and collinear singularities. Because we restrict ourselves to the operator $\hat{O}_7$, the photon energy spectrum is finite after the standard renormalization procedure is performed.

At leading order in perturbation theory the kinematics of the decay $b \to s + \gamma$ is very simple: the photon and the (massless) strange quark are back-to-back with equal energy and momentum. Such a simple kinematics implies that the energy spectrum of the photon at this order is given by a delta-function $\sim \delta(1-z)$. At order $\mathcal{O}(\alpha_s)$, the gluon radiation smears the photon energy spectrum and a radiative tail down to $E_\gamma = 0$ appears. Therefore, the $\mathcal{O}(\alpha_s^2)$ correction to the photon energy spectrum is the first non-trivial QCD correction to the radiative tail.

For the physical observables studied in this paper, we only need to compute the $\mathcal{O}(\alpha_s^2)$ corrections to the photon energy spectrum away from the maximal value of the photon energy $E_\gamma < E_{\rm max}$. As a consequence, we do not compute the two-loop virtual corrections to $b \to s + \gamma$ since they only contribute for $z = 1$. Therefore, to calculate the $\mathcal{O}(\alpha_s^2)$ corrections to the photon energy spectrum in $b \to X_s + \gamma$ for $z < 1$, we have to consider processes with up to two gluons or a quark-antiquark pair in addition to the strange quark and the photon in the final state, as well as the virtual corrections to single gluon emission. To perform the calculation, we use the techniques introduced in [35, 36, 37]. The idea is to apply the optical theorem to $b \to b$ transition amplitude in the presence of the constraint on the photon energy. The constraint, $\delta(E_\gamma - zE_{\rm max})$, is treated as the on-shell condition for a "fake" particle. Then, evaluation of the complicated integrals over multi-particle phase-space, e.g. $b \to s + \gamma + g + g$ is simplified by multiloop integration technology [38]. To solve the integration-by-parts identities, we use the algorithm of Ref. [39] implemented in [40].

It is convenient to present the result for the photon energy spectrum in a normalized form, dividing the differential rate $\mathrm{d}\Gamma/\mathrm{d}z$ by the total width for $B \to X_s + \gamma$. This ratio is convenient because its integral over the fraction of the photon energy z is equal to one to all orders in the strong coupling constant α_s:

$$\int\limits_0^1 \frac{1}{\Gamma}\frac{d\Gamma}{dz}dz = 1. \tag{3}$$

Note that because of Eq.(3), we can restore the $\delta(1-z)$ terms in the normalized photon energy spectrum without explicit computation of the two-loop virtual corrections.

Our result for the photon energy spectrum reads:

$$\frac{1}{\Gamma}\frac{d\Gamma}{dz} = \delta(1-z) + \left(\frac{\alpha_s}{\pi}\right) C_F F^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 C_F F^{(2)}, \tag{4}$$

where

$$F_2 = C_F F^{(2,\rm a)} + C_A F^{(2,\rm na)} + T_R n_f F^{(2,\rm cf)}, \tag{5}$$

[1] Because of the Fermi motion of the heavy quark inside the B-meson, the photon energy spectrum extends beyond the point $m_b/2$ all the way up to $E_{\rm max} = M_B/2$, where M_B is the mass of the B-meson.

α_s is the $\overline{\rm MS}$ coupling constant renormalized at the b-quark pole mass, $C_F = 4/3,\ C_A = 3, T_R = 1/2$. Also, $n_f = 4$ is the number of light fermion flavors.

The coefficient functions read:

$$F^{(1)} = \left\{ -\frac{31}{12}\delta(1-z) - \left[\frac{\ln(1-z)}{1-z}\right]_+ - \frac{7}{4}\left[\frac{1}{1-z}\right]_+ - \frac{z+1}{2}\ln(1-z) + \frac{7+z-2z^2}{4}\right\}, \tag{6}$$

and

$$\begin{aligned}
F^{(2,\mathrm{a})} &= S_{\mathrm{a}}\delta(1-z) + \frac{1}{2}\left[\frac{\ln^3(1-z)}{1-z}\right]_+ + \frac{21}{8}\left[\frac{\ln^2(1-z)}{1-z}\right]_+ + \left(-\frac{\pi^2}{6} + \frac{271}{48}\right)\left[\frac{\ln(1-z)}{1-z}\right]_+ \\
&+ \left(\frac{425}{96} - \frac{\pi^2}{6} - \frac{\zeta(3)}{2}\right)\left[\frac{1}{1-z}\right]_+ + \frac{4z-4z^2+1+z^3}{2(z-1)}\left[\mathrm{Li}_3\left(\frac{z}{2-z}\right) - \mathrm{Li}_3\left(-\frac{z}{2-z}\right) - 2\mathrm{Li}_3\left(\frac{1}{2-z}\right) + \frac{\zeta(3)}{4}\right] \\
&+ \left[\frac{z^3-2z^2+2z-3}{2(z-1)}\ln(1-z) - \frac{-140z^4+219z^3-124z^2+28z+27z^5+9z^6+z^8-6z^7-6}{12z(z-1)^3}\right]\mathrm{Li}_2(z-1) \\
&-2(z-1)^2\mathrm{Li}_3(z-1) + \left[\frac{2z^3-9z^2-2z+11}{4(z-1)}\ln(1-z) - \frac{-27z^2+8z^6-9+21z-3z^3+64z^4-46z^5}{12z(z-1)^3}\right]\mathrm{Li}_2(1-z) \\
&-\frac{-17z^2+4z+4z^3+11}{4(z-1)}\mathrm{Li}_3(1-z) - \frac{2z^3+13-9z^2}{4(z-1)}\mathrm{Li}_3(z) + \frac{4z-4z^2+1+z^3}{6(z-1)}\ln^3(2-z) \\
&+ \left[-\frac{4z-4z^2+1+z^3}{2(z-1)}\ln^2(1-z) - \frac{-140z^4+219z^3-124z^2+28z+27z^5+9z^6+z^8-6z^7-6}{12z(z-1)^3}\ln(1-z)\right. \\
&\left.-\frac{4z-4z^2+1+z^3}{z-1}\frac{\pi^2}{12}\right]\ln(2-z) + \frac{z^3-2z^2+2z+1}{4z}\ln^3(1-z) + \frac{z^5-3z^4+5z^3+7z^2+5z-9}{24z}\ln^2(1-z) \\
&+ \left[-\frac{z^2+8z-11}{8(z-1)}\ln^2(1-z) - \frac{-27z^2+8z^6-9+21z-3z^3+64z^4-46z^5}{12z(z-1)^3}\ln(1-z)\right]\ln(z) \\
&+ \left[(-z^2+z-3)\frac{\pi^2}{12} - \frac{4z^5+151z+2z^4-48z^2-41z^3-36}{48z(z-1)}\right]\ln(1-z) - \frac{(z-2)(z^4-z^3-11z^2+13z+3)}{z}\frac{\pi^2}{72} \\
&+\frac{z^3-11z^2-2z+18}{4(z-1)}\zeta(3) - \frac{8z^4-244z^3+175z^2+598z-569}{96(z-1)}\ ,
\end{aligned} \tag{7}$$

$$\begin{aligned}
F^{(2,\mathrm{na})} &= S_{\mathrm{na}}\delta(1-z) + \frac{11}{8}\left[\frac{\ln^2(1-z)}{1-z}\right]_+ + \left(\frac{\pi^2}{12} + \frac{95}{144}\right)\left[\frac{\ln(1-z)}{1-z}\right]_+ + \left(\frac{\zeta(3)}{4} - \frac{905}{288} + \frac{17\pi^2}{72}\right)\left[\frac{1}{1-z}\right]_+ \\
&-\frac{4z-4z^2+1+z^3}{4(z-1)}\left[\mathrm{Li}_3\left(\frac{z}{2-z}\right) - \mathrm{Li}_3\left(-\frac{z}{2-z}\right) - 2\mathrm{Li}_3\left(\frac{1}{2-z}\right) + \frac{\zeta(3)}{4}\right] + (z-1)^2\mathrm{Li}_3(z-1) \\
&+ \left[-\frac{z^3-2z^2+2z-3}{4(z-1)}\ln(1-z) + \frac{-140z^4+219z^3-124z^2+28z+27z^5+9z^6+z^8-6z^7-6}{24z(z-1)^3}\right]\mathrm{Li}_2(z-1) \\
&+ \left[\frac{z(3-z)}{4}\ln(1-z) + \frac{(1+z)(2z^4-29z^3+73z^2-57z+15)}{24(z-1)^3}\right]\mathrm{Li}_2(1-z) + \frac{4z-4z^2+1+z^3}{4(z-1)}\mathrm{Li}_3(z) \\
&+\frac{(z-3)z}{2}\mathrm{Li}_3(1-z) - \frac{4z-4z^2+1+z^3}{12(z-1)}\ln^3(2-z) + \left[\frac{4z-4z^2+1+z^3}{4(z-1)}\ln^2(1-z)\right. \\
&\left.+\frac{-140z^4+219z^3-124z^2+28z+27z^5+9z^6+z^8-6z^7-6}{24z(z-1)^3}\ln(1-z) + \frac{4z-4z^2+1+z^3}{z-1}\frac{\pi^2}{24}\right]\ln(2-z) \\
&+\frac{(1+z)(2z^4-29z^3+73z^2-57z+15)}{24(z-1)^3}\ln(1-z)\ln(z) - \frac{(z-1)^2}{8}\ln^3(1-z) - \frac{(z+2)(z^3-5z^2+9z-35)}{48}\ln^2(1-z)
\end{aligned}$$

$$+\left[(z^2-z+3)\frac{\pi^2}{24}+\frac{6z^5+72-392z^3+51z^4+219z^2+92z}{144z(z-1)}\right]\ln(1-z)+\frac{z^5-3z^4-3z^3+34z^2-24z+3}{z}\frac{\pi^2}{144}$$
$$-\frac{z^3-10z^2+6z+7}{8(z-1)}\zeta(3)+\frac{12z^4-754z^3+1191z^2+264z-761}{288(z-1)}\,, \qquad (8)$$

$$F^{(2,\mathrm{nf})}=S_{\mathrm{nf}}\delta(1-z)-\frac{1}{2}\left[\frac{\ln^2(1-z)}{1-z}\right]_+-\frac{13}{36}\left[\frac{\ln(1-z)}{1-z}\right]_++\left(-\frac{\pi^2}{18}+\frac{85}{72}\right)\left[\frac{1}{1-z}\right]_+$$
$$+\frac{z^2-3}{6(z-1)}\mathrm{Li}_2(1-z)+\frac{z^2-3}{6(z-1)}\ln(1-z)\ln(z)-\frac{1+z}{4}\ln^2(1-z)-\frac{6z^3-25z^2-z-18}{36z}\ln(1-z)$$
$$-(1+z)\frac{\pi^2}{36}+\frac{-49+38z^2-55z}{72}\,. \qquad (9)$$

Here, $\mathrm{Li}_3(x)=\int_0^x \mathrm{d}x_1 \mathrm{Li}_2(x_1)/x_1$, $\zeta(3)$ is the Riemann zeta-function and $[\ln^n(1-x)/(1-x)]_+$ are the plus-distributions defined in the standard way. The constants S_i take the following values $S_\mathrm{a}=1.216$, $S_{\mathrm{na}}=-4.795$, $S_{\mathrm{nf}}=49/24+\pi^2/8-2\zeta(3)/3\approx 2.474$. In addition, for simplicity, we have set the renormalization and the factorization scales in Eq.(4) equal to the b-quark pole mass m_b. The total width of the radiative transition $B\to X_s+\gamma$ is known through $\mathcal{O}(\alpha_s)$. It reads:

$$\Gamma=\Gamma^{(0)}\left(1+\frac{\alpha_s}{\pi}C_F\left(\frac{4}{3}-\frac{\pi^2}{3}\right)+\mathcal{O}(\alpha_s^2)\right), \qquad (10)$$

where $\Gamma^{(0)}=\dfrac{\alpha G_F^2}{32\pi^4}|V_{tb}V_{ts}^*|^2C_7^2(m_b)\; m_b^3\overline{m}^2(m_b)$.

Our result for the photon energy spectrum Eq.(4) can be compared to partial results that exist in the literature. The correction to the spectrum proportional to the number of massless flavors n_f agrees with the result in [24]. In addition, the terms in the photon energy spectrum that develop $[\ln^n(1-z)/(1-z)]_+$, $n=0\ldots3$ singularities for $z\to1$ agree with [32]. We note that the photon energy spectrum computed in this paper provides an additional check of the constant D_2 that was extracted in [41] from our previous calculation of the perturbative heavy quark fragmentation function [42].

III. ANALYSIS OF THE PERTURBATIVE SPECTRUM

In this Section we discuss the properties of the perturbative photon energy spectrum Eq.(4) and compare it with various approximations available in the literature. For the illustration we use $\alpha_s=\alpha_s(m_b)=0.22$ for all plots in this Section. A choice of α_s that is more appropriate for observables when a cut on the photon energy is applied, is discussed in the next Section.

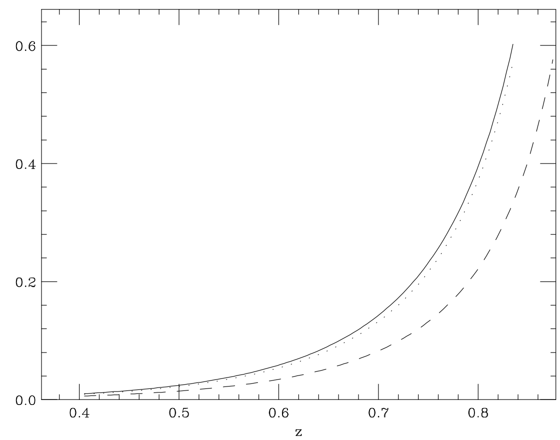

FIG. 1: The normalized perturbative spectrum Eq.(4) in $b\to s+\gamma$ retaining $\mathcal{O}(\alpha_s^2)$ (solid), BLM (dots) and $\mathcal{O}(\alpha_s)$ terms.

An overview of the photon energy spectrum is given in Fig.1, where the $\mathcal{O}(\alpha_s)$, the full $\mathcal{O}(\alpha_s^2)$ and the BLM approximation to the spectrum are compared. It follows that when the renormalization scale is chosen equal to the mass of the b-quark, the bulk of the $\mathcal{O}(\alpha_s^2)$ correction is provided by the BLM terms and that the non-BLM corrections are moderate.

As we mentioned in the Introduction, the BLM corrections require a special treatment; given their magnitude, the naive interpretation of Fig.1 results in a large perturbative uncertainty on the photon energy spectrum. Fortunately, the effects of the BLM corrections have been resummed to all orders in a way consistent with the Wilsonian approach to the operator product expansion (OPE) for $b\to X_s+\gamma$ [29]. Since the impact of the BLM corrections on the photon energy spectrum is well-understood, we can concentrate in the following analysis on the effect of the non-BLM corrections. This is the new contribution to the photon energy spectrum derived in this paper.

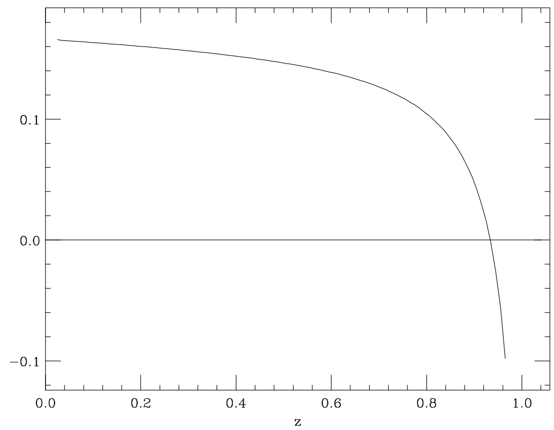

FIG. 2: The non-BLM $\mathcal{O}(\alpha_s^2)$ correction to the photon energy spectrum relative to the $\mathcal{O}(\alpha_s)$ correction.

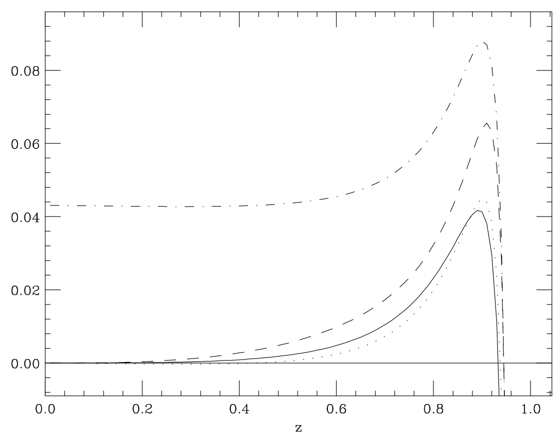

FIG. 3: The non-BLM $\mathcal{O}(\alpha_s^2)$ correction (solid), its $z \to 1$ approximation (dots-dashes), the $z \to 1$ approximation minus its value for $z = 0$ (dots) and the $z \to 1$ asymptotics times z^3 (dashes).

The magnitude of the non-BLM part of the $\mathcal{O}(\alpha_s^2)$ correction is shown in Fig.2, where it is plotted relative to the photon energy spectrum through $\mathcal{O}(\alpha_s)$. The relative size of the non-BLM $\mathcal{O}(\alpha_s^2)$ correction is within $\pm 10\%$, approximately.

It is interesting to check if the terms $[\ln^n(1-z)/(1-z)]_+$, $n = 0 \ldots 3$ in Eq.(4) that are singular in $z \to 1$ limit, provide a good approximation to non-BLM corrections; Figs.3,4 illustrate this. We see that the singular terms do not furnish a good approximation to the exact result for moderately large values of z; for example, for $z = 0.8$, the $z \to 1$ approximation overestimates the exact result by a factor of three. This comparison suggests that based on the previously known large z behavior of the non-BLM piece [32], one could not have computed its contribution to observables, unless very high value of the cut on the photon energy is in place.

Although the $z \to 1$ approximation alone is insufficient to adequately approximate the non-BLM correction, it is possible to improve the quality of the approximation by

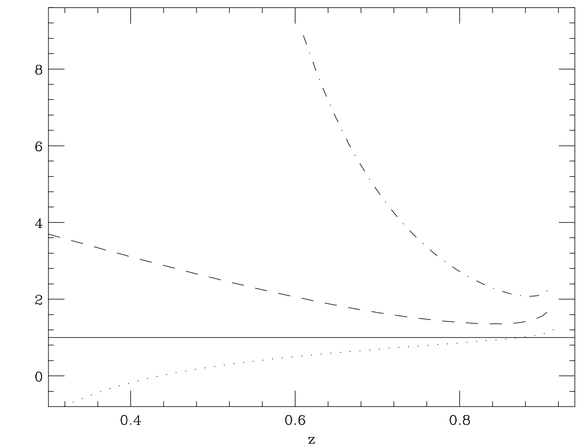

FIG. 4: Same as Fig.3. Each curve is normalized to the non-BLM $\mathcal{O}(\alpha_s^2)$ correction.

combining the $z \to 1$ approximation with the $z \to 0$ behavior of the photon energy spectrum. It is easy to see that for $\hat{O}_7$ operator the photon energy spectrum should vanish as z^3, for $z \to 0$. Hence, we can modify the $z \to 1$ asymptotics by multiplying it by z^3. This improves the quality of the approximation; for $z = 0.8$, the modified $z \to 1$ approximation overestimates the exact result by a factor 1.3. However, the best description of the exact result is obtained by subtracting from $z \to 1$ asymptotics its value at $z = 0$. While this approximation is hard to justify theoretically, it gives quite an accurate description of the spectrum for moderately large values of z.

IV. APPLICATIONS

In this Section, we study the numerical implications of the $\mathcal{O}(\alpha_s^2)$ correction to the photon energy spectrum derived in this paper. As noted in the Introduction, high-quality experimental results on $B \to X_s + \gamma$ become available from the B-factories. In particular the branching fraction of $B \to X_s + \gamma$ and the average energy of the photon in the presence of a cut $E_\gamma > E_{\rm cut}$ are measured precisely.

The applicability of the pure perturbative description of the photon energy spectrum depends upon the exact value of $E_{\rm cut}$ which, in turn, defines the degree of "inclusiveness" of the process. If $E_{\rm cut}$ is high, the usual OPE breaks down and the resummation of leading twist effects in the OPE leads to the appearance of the non-perturbative shape function. By decreasing $E_{\rm cut}$ it becomes possible to describe the non-perturbative component of the photon energy spectrum by means of the local OPE. It is generally believed that for $E_{\rm cut}$ as low as 1.8 GeV, we are already in the OPE regime. Fortunately, Belle [1] published results on the branching fraction and the first two moments for precisely this value of the cut on the photon energy $E_\gamma > 1.815$ GeV.

To proceed further, we have to specify the numeri-

cal value of the b-quark mass. Although our result for the spectrum is derived in the pole scheme, it is clear on general grounds that the pole mass can not appear in short-distance quantities; instead, a consistent application of OPE leads to short-distance, low-scale quark masses that enter the perturbative calculations. In principle, the transformation from one scheme to another should be done in a self-consistent way, including the proper treatment of non-perturbative effects. However, as we will see, the contribution of the non-BLM $\mathcal{O}(\alpha_s^2)$ correction to observables is relatively small and, therefore, a simple estimate suffices. For this reason, instead of the pole b-quark mass we use in our formulas $m_b = 4.6$ GeV which roughly approximates the kinetic b-quark mass normalized at $\mu = 1$ GeV [43]. For the two applications that we consider below, the exact value of the b quark mass mostly affects the z integration region, through $z > z_{\rm cut} = 2E_{\rm cut}/m_b$. In addition, we note that the scale at which the strong coupling constant has to be evaluated in the numerical estimates has to be smaller than the mass of the b-quark. This follows from the fact that, for a given value of $E_{\rm cut}$, the maximal value of the invariant mass of the hadronic system in $b \to X_s + \gamma$ is $\sqrt{m_b(m_b - 2E_{\rm cut})} \sim 2$ GeV for $E_{\rm cut} \sim 2$ GeV. This implies that for typical values of the cut on the photon energy the appropriate value for the scale at which α_s should be evaluated is $\mu \sim 1.5 - 2$ GeV. Having made these preliminary remarks, we are ready to present our numerical estimates.

First, we consider the fraction of events that contain a photon with energy above the cut $E_{\rm cut}$. The second order QCD corrections shift this fraction by:

$$\delta R(E_{\rm cut}) = -\left(\frac{\alpha_s}{\pi}\right)^2 \Delta_R(z_{\rm cut}),$$
$$\Delta_R(z_{\rm cut}) = C_F \int\limits_0^{z_{\rm cut}} {\rm d}z F^{(2)}(z), \qquad (11)$$

where we have used Eq.(3) to express the integral over $z > z_{\rm cut}$ through the lower part of the photon energy spectrum. As we explained previously, the BLM corrections are taken into account in the estimates for the fraction of events that exist in the literature. Therefore, we disregard the BLM corrections and present the contribution of the non-BLM corrections to Eq.(11) in Fig.5. With our choice of the b-quark mass and for $E_{\rm cut} = 1.8$ GeV, we find $z_{\rm cut} = 0.78$ and $\Delta_R = 0.50$ which, depending on the value of $\alpha_s = 0.3 \div 0.35$, translates into:

$$\delta R_{\rm non-BLM} = -(5.5 \pm 0.5) \times 10^{-3}.$$

The predictions for the fraction of events with the photon energy larger than 1.8 GeV that are available in the literature are: $R = 0.952^{+0.013}_{-0.029}$ [23], $R = 0.958^{+0.013}_{-0.029}$ [13], $R = 0.95 \pm 0.01$ [30] and $R = 0.89^{+0.06}_{-0.07}$ [44], where in the last result only perturbative uncertainty is displayed. We see that the second order non-BLM QCD corrections

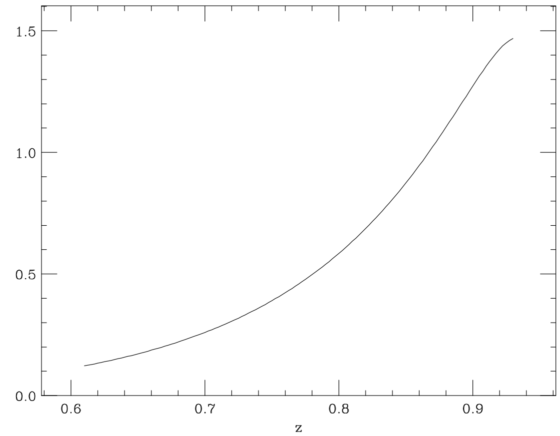

FIG. 5: The function $\Delta_R(z)$ Eq.(11) for an experimentally relevant range of z_{cut}. Only the non-BLM corrections are used in the calculation.

change the fraction of events with $E_\gamma > E_{\rm cut}$ by less than a percent and are about half of the uncertainty assigned to $R(E_{\rm cut})$ in [30].

The next observable we consider is the first moment of the photon energy spectrum $\langle E_\gamma \rangle$ for $E_\gamma > E_{\rm cut}$. It is straightforward to derive the correction to the average photon energy when the second order QCD corrections are applied. The result reads:

$$\delta\langle E_\gamma\rangle = -\frac{m_b}{2}\left(\frac{\alpha_s}{\pi}\right)^2 \Delta_E(z_{\rm cut}),$$
$$\Delta_E(z_{\rm cut}) = C_F \int\limits_{z_{\rm cut}}^1 {\rm d}z(1-z)F^{(2)}(z)$$
$$+C_F^2 \int\limits_{z_{\rm cut}}^1 {\rm d}z(1-z)F^{(1)}(z) \int\limits_0^{z_{\rm cut}} {\rm d}y F^{(1)}(y). \qquad (12)$$

The last term in Eq.(12) comes from the interference of $\mathcal{O}(\alpha_s)$ corrections to the numerator and denominator in the expression for $\langle E_\gamma \rangle$. We disregard the BLM corrections and compute Δ_E as a function of $z_{\rm cut}$. The result is shown in Fig.6. For $E_{\rm cut} = 1.8$ GeV, we get $z_{\rm cut} = 0.78$ and obtain $\Delta_E = -0.68$. This leads to:

$$\delta\langle E_\gamma\rangle_{\rm non-BLM} = (1.8 \pm 0.3) \times 10^{-2}\ {\rm GeV}, \qquad (13)$$

depending on the value of the strong coupling constant.

The current most accurate measurement of the average energy in $B \to X_s + \gamma$ comes from Belle collaboration [1] and reads $\langle E_\gamma\rangle_{E_\gamma > 1.8\ {\rm GeV}} = (2.292 \pm 0.026 \pm 0.034)$ GeV. The theoretical estimates for this observable that are available in the literature are: $\langle E_\gamma \rangle = 2.27^{+0.05}_{-0.07}$ GeV [44], where only the uncertainty associated with uncalculated higher order corrections is displayed, and $\langle E_\gamma \rangle = 2.312$ GeV [29] (no explicit estimate of the uncertainty has been given in that reference). Our result Eq.(13) shows that the central values in these predictions should be shifted approximately by one percent. In addition, as

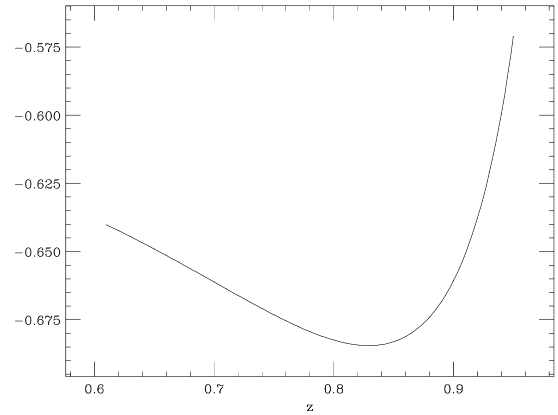

FIG. 6: The function $\Delta_E(z)$ Eq.(12) for an experimentally relevant range of z_{cut}. Only the non-BLM corrections are used in the calculation.

the result of our calculation, the error bars for $\langle E_\gamma \rangle$ associated with higher order QCD corrections given in [44] should be reduced to at most a percent.

We note that although a percent shift in the central value for $\langle E_\gamma \rangle$ is small, it can not be considered negligible given the current precision achieved in B-physics. In particular, the average photon energy plays a central role in constraining the mass of the b-quark from B-decays. The correction that we have computed in Eq.(13) results in *approximately* -30 MeV shift in the short-distance low-scale mass. The value of this shift can not be neglected because it roughly equals the uncertainty in the value of the b-quark mass that was extracted from recent fits to semi-leptonic and radiative B-decays [45].

V. CONCLUSIONS

The knowledge of the photon energy spectrum in $B \to X_s + \gamma$ is required to relate the experimental measurements of the branching fraction $\mathrm{Br}[B \to X_s + \gamma]$ with a lower cut on the photon energy and the fully inclusive theoretical calculations. This observable is of particular interest because it is sensitive to physics beyond the Standard Model and represents the cleanest observable for theoretical computations. In addition, the photon energy spectrum is used to test our understanding of strong interactions and to extract fundamental parameters of heavy flavor physics.

In this paper the analytic computation of the $\mathcal{O}(\alpha_s^2)$ corrections to the photon energy spectrum is presented. We have restricted ourselves to the contribution of the operator $\hat{O}_7$. Partial analytical results on the $\mathcal{O}(\alpha_s^2)$ corrections to the photon energy spectrum that have been published previously are confirmed. On the other hand, we have shown that the terms that are singular in $z \to 1$ limit do not furnish a good numerical approximation to $\mathcal{O}(\alpha_s^2)$ corrections for $z < 0.95$.

To evaluate the numerical impact of the $\mathcal{O}(\alpha_s^2)$ corrections, we separate the BLM and the non-BLM pieces. While the BLM corrections have been computed and analyzed previously, the non-BLM component at $\mathcal{O}(\alpha_s^2)$ represents the new result of this paper. The impact of this new result on the fraction of events with photons having energies larger than E_{cut} and on the average energy of the photons in $B \to X_s + \gamma$ is in the 1% range. This new contribution is comparable in size to both the accuracy of the experimental measurements and the uncertainty assigned to these observables in previous evaluations.

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